Verify for example 2, p 77.

```
DSolve[y''[x] - 2 y'[x] + y[x] == 0, y, x]
```

$\{\{y \to \text{Function}[\{x\}, e^x C[1] + e^x x C[2]]\}\}$

verified.

2 - 8 Basis of solutions. Wronskian
Find the Wronskian. Show linear independence by using quotients and confirm it by theorem 2, p. 75.

3. $e^{-0.4 x}$, $e^{-2.6 x}$

```
ClearAll["Global`*"]
```

$\text{Wronskian}\left[\left\{e^{-0.4 x}, e^{-2.6 x}\right\}, x\right]$

$-2.2 e^{-3. x}$

$\text{Solve}\left[e^{-0.4 x} == a e^{-2.6 x}, a\right]$

$\left\{\left\{a \to 1. e^{2.2 x}\right\}\right\}$

The factor that joins the two quantities is not constant. Therefore the two quantities are linearly independent.

5. x^3, x^2

```
ClearAll["Global`*"]
```

$\text{Wronskian}\left[\left\{x^3, x^2\right\}, x\right]$

$-x^4$

$\text{Solve}\left[x^3 == a x^2, a\right]$

$\{\{a \to x\}\}$

The factor that joins the two quantities is not constant. Therefore the two quantities are linearly independent.

7. Cosh[a x], Sinh[a x]

```
ClearAll["Global`*"]
Wronskian[{Cosh[a x], Sinh[a x]}, x]
```

a

```
Solve[Cosh[a x] == r Sinh[a x], r]
{{r → Coth[a x]}}
```

The factor that joins the two quantities is not constant. Therefore the two quantities are linearly independent.

9 - 15 ODE for given basis. Wronskian. IVP.
(a) Find a second-order homogeneous linear ODE for which the given functions are solutions. (b) Show linear independence by the Wronskian. (c) Solve the initial value problem.

9. Cos[5 x], Sin[5 x], y[0] = 3, y'[0] = -5

```
ClearAll["Global`*"]
```

Part (a) I am assuming here that I need to use both given parts at the second order in the ODE I am looking for.

```
h[x_] := Cos[5 x]

D[h[x], x]
-5 Sin[5 x]

D[h[x], {x, 2}]
-25 Cos[5 x]

DSolve[y''[x] - (-25 Cos[5 x]) == 0, y, x]
{{y → Function[{x}, C[1] + x C[2] + Cos[5 x]]}}

g[x_] := Sin[5 x]

D[g[x], x]
5 Cos[5 x]

D[g[x], {x, 2}]
-25 Sin[5 x]
```

So if I were to add g”[x] and h”[x] together I would get -25(Sin[5x] +Cos[x]), and if I want the overall thing to equal zero, I just have to add 25y. Therefore

How about $y'' + 25\, y == 0$

```
DSolve[y''[x] + 25 y[x] == 0, y[x], x]
{{y[x] → C[1] Cos[5 x] + C[2] Sin[5 x]}}
```

Part (b)

```
Wronskian[{Cos[5 x], Sin[5 x]}, x]
```

```
5
```

The Wronskian is non-zero, implying that the two functions are linearly independent.

Part (c). Using the equation I already found for part (a), I put in the initial values.

```
vwt = {y''[x] + 25 y[x] == 0, y[0] == 3, y'[0] == -5}
{25 y[x] + y''[x] == 0, y[0] == 3, y'[0] == -5}
```

```
vct = DSolve[vwt, y, x]
```

```
{{y → Function[{x}, 3 Cos[5 x] - Sin[5 x]]}}
```

```
Simplify[vwt /. vct]
{{True, True, True}}
```

11. $e^{-2.5 x}$ Cos[0.3 x], $e^{-2.5 x}$ Sin[0.3 x], y[0] = 3, y'[0] = -7.5

```
ClearAll["Global`*"]
```

Part (a) An ugly alternative to the text answer's equation. From something I read, it seems possible that using the characteristic equation corresponding to the problem functions could make things smooth and yield the text answer.

```
g[x_] = e^(-2.5 x) (Cos[0.3 x])
e^(-2.5 x) Cos[0.3 x]
```

```
Simplify[D[g[x], x]]
e^(-2.5 x) (-2.5 Cos[0.3 x] - 0.3 Sin[0.3 x])
```

I mark the 2nd order first half.

```
Simplify[D[g[x], {x, 2}]]
```

```
e^(-2.5 x) (6.16 Cos[0.3 x] + 1.5 Sin[0.3 x])
```

```
h[x_] = e^(-2.5 x) (Sin[0.3 x])
e^(-2.5 x) Sin[0.3 x]
```

```
Simplify[D[h[x], x]]
e^(-2.5 x) (0.3 Cos[0.3 x] - 2.5 Sin[0.3 x])
```

I mark the 2nd order second half.

```
Simplify[D[h[x], {x, 2}]]
```

```
e^(-2.5 x) (-1.5 Cos[0.3 x] + 6.16 Sin[0.3 x])
```

I just add the two parts together

```
eqn = ⅇ^(-2.5 x) (6.16 (Cos[0.3 x] + Sin[0.3 x]) + 1.5 (Sin[0.3 x] - Cos[0.3 x]))
```

$$e^{-2.5\,x}\,(1.5\,(-\mathrm{Cos}[0.3\,x] + \mathrm{Sin}[0.3\,x]) + 6.16\,(\mathrm{Cos}[0.3\,x] + \mathrm{Sin}[0.3\,x]))$$

And then solve the resulting ODE (which is linear and homogeneous)

```
sol = DSolve[y''[x] == eqn, y, x]
```

$$\{\{y \to \mathrm{Function}[\{x\},\, C[1] + x\,C[2] + e^{-2.5\,x}\,((1. - 4.16334\times10^{-17}\,i)\,\mathrm{Cos}[0.3\,x] + (1. - 4.16334\times10^{-17}\,i)\,\mathrm{Sin}[0.3\,x])]\}\}$$

And I need to apply a light chop to get rid of some small fragments attached by machine precision border slop, and assign values to constants. Afterwards, I see that I have recovered the original functions. Due to this success, the cyan cell above holds the proposed solution.

```
Chop[sol, 10^-16] /. {C[1] → 0, C[2] → 0}
```

$$\{\{y \to \mathrm{Function}[\{x\},\, 0 + x\,0 + e^{-2.5\,x}\,(1.\,\mathrm{Cos}[0.3\,x] + 1.\,\mathrm{Sin}[0.3\,x])]\}\}$$

Part (b) The answer for the Wronskian matches the text. The Wronskian is non-zero, implying that the two functions are linearly independent.

```
Wronskian[{ⅇ^(-2.5 x) Cos[0.3 x], ⅇ^(-2.5 x) Sin[0.3 x]}, x]
```

$$0.3\,e^{-5.\,x}$$

Part (c) This is where the initial value problem is solved, amounting to a tailoring of the cyan function.

```
solivp = DSolve[{y''[x] == eqn, y[0] == 3, y'[0] == -7.5}, y, x]
```

$$\{\{y \to \mathrm{Function}[\{x\},\, (-5.3 - 9.15934\times10^{-17}\,i)\,e^{-2.5\,x}\,((-0.377358 - 1.33393\times10^{-18}\,i)\,e^{2.5\,x} + (1. + 0.\,i)\,e^{2.5\,x}\,x - (0.188679 - 1.11161\times10^{-17}\,i)\,\mathrm{Cos}[0.3\,x] - (0.188679 - 1.11161\times10^{-17}\,i)\,\mathrm{Sin}[0.3\,x])]\}\}$$

The pink cell below holds the ivp solution candidate.

```
Chop[solivp, 10^-16] /. {C[1] → 0, C[2] → 0}
```

$$\{\{y \to \mathrm{Function}[\{x\},\, -5.3\,e^{-2.5\,x}\,(-0.377358\,e^{2.5\,x} + 1.\,e^{2.5\,x}\,x - 0.188679\,\mathrm{Cos}[0.3\,x] - 0.188679\,\mathrm{Sin}[0.3\,x])]\}\}$$

Now testing the ivp solution candidate at x=0.

```
soli = -5.300000000000001` e^(-2.5` x)
    (-0.37735849056603776` e^(2.5` x) + 1.` e^(2.5` x) x - 0.18867924528301877`
      Cos[0.3` x] - 0.18867924528301877` Sin[0.3` x]) /. x → 0
```

```
3.
```

I need to differentiate the candidate in order to test the derivative.

```
D[-5.300000000000001` e^(-2.5` x)
   (-0.37735849056603776` e^(2.5` x) + 1.` e^(2.5` x) x - 0.18867924528301877`
     Cos[0.3` x] - 0.18867924528301877` Sin[0.3` x]), x]
```

```
13.25 e^(-2.5 x)
   (-0.377358 e^(2.5 x) + 1. e^(2.5 x) x - 0.188679 Cos[0.3 x] - 0.188679 Sin[0.3 x]) -
  5.3 e^(-2.5 x) (0.0566038 e^(2.5 x) + 2.5 e^(2.5 x) x -
     0.0566038 Cos[0.3 x] + 0.0566038 Sin[0.3 x])
```

Now testing the ivp solution candidate derivative form at x=0.

```
13.250000000000002` e^(-2.5` x)
    (-0.37735849056603776` e^(2.5` x) + 1.` e^(2.5` x) x - 0.18867924528301877`
      Cos[0.3` x] - 0.18867924528301877` Sin[0.3` x]) -
   5.300000000000001` e^(-2.5` x) (0.05660377358490565` e^(2.5` x) +
      2.5` e^(2.5` x) x - 0.05660377358490563` Cos[0.3` x] +
      0.05660377358490563` Sin[0.3` x]) /. x → 0
```

```
-7.5
```

The ivp solution candidate passes the iv tests, yielding in green above, the text answer for Wronskian value, and in lilac, the prescribed initial values. An ugly duckling it may be, but it works. For reference, the text answer for the second order ODE is y”+5y’+6.34=0 and for the ivp sol’n it is 3 $e^{-2.5}$ Cos[0.3 x].

13. 1, e^{-2x}, y[0] = 1, y'[0] = -1

```
In[2]:= ClearAll["Global`*"]
```

Part (a) Going for the second function. The first one, a constant, can be added in with a constant after the **DSolve** step.

```
In[3]:= hax = e^(-2 x)
Out[3]= e^(-2 x)

In[4]:= haxd1 = D[hax, x]
Out[4]= -2 e^(-2 x)
```

```
In[5]:= haxd2 = D[hax, {x, 2}]
```

Out[5]= $4\, e^{-2x}$

```
In[6]:= eqn = 4 e^(-2 x)
```

Out[6]= $4\, e^{-2x}$

Testing the solution to see if **DSolve** can use it to recover the starting functions.

```
In[7]:= sol = DSolve[y''[x] == eqn, y, x]
```

Out[7]= $\{\{y \to \text{Function}[\{x\}, e^{-2x} + C[1] + x\, C[2]]\}\}$

With judicious constant choice, the functions are recouped.

```
sol1 = sol /. {C[1] → 1, C[2] → 0}
```

$\{\{y \to \text{Function}[\{x\}, e^{-2x} + 1 + x\, 0]\}\}$

Part (b) The answer for the Wronskian matches the text. The Wronskian is non - zero, implying that the two functions are linearly independent.

```
Wronskian[{1, e^(-2 x)}, x]
```

$-2\, e^{-2x}$

Part (c) This is where the initial value problem is solved, amounting to a tailoring of the cyan function.

```
soltail = DSolve[{y''[x] == 4 e^(-2 x), y[0] == 1, y'[0] == -1}, y, x]
```

$\{\{y \to \text{Function}[\{x\}, e^{-2x}(1 + e^{2x}\, x)]\}\}$

A proposed function is shown.

```
Simplify[e^(-2 x) (1 + e^(2 x) x)]
```

$e^{-2x} + x$

The proposed function succeeds in handling the first initial value.

```
% /. x → 0
```

1

The derivative is obtained for the second test.

```
D[e^(-2 x) + x, x]
```

$1 - 2\, e^{-2x}$

The proposed function succeeds in handling the second initial value.

```
% /. x → 0
```

```
-1
```

However, the proposed function is not the one found by the text answer in the appendix.

```
PossibleZeroQ[(e^(-2 x) + x) - 0.5 (1 + e^(-2 x))]
```

```
False
```

For reference, the text answer for the second order ODE is y”+2y’=0 and for the ivp sol’n it is $0.5(1+e^{-2x})$.

15. Cosh[1.8 x], Sinh[1.8 x], y[0] = 14.20, y'[0] = 16.38

```
ClearAll["Global`*"]
```

Part (a). As in the above problems, I cobble together a second order sum and **DSolve** it.

```
D[Cosh[1.8 x], x]
```

```
1.8 Sinh[1.8 x]
```

```
D[Sinh[1.8 x], x]
```

```
1.8 Cosh[1.8 x]
```

```
D[Cosh[1.8 x], {x, 2}]
```

```
3.24 Cosh[1.8 x]
```

```
D[Sinh[1.8 x], {x, 2}]
```

```
3.24 Sinh[1.8 x]
```

```
eqn = 3.24 (Cosh[1.8 x] + Sinh[1.8 x])
```

```
3.24 (Cosh[1.8 x] + Sinh[1.8 x])
```

The second order equation I am using for y”[x] does not match the text’s.

```
sol = DSolve[y''[x] == eqn, y, x]
```

```
{{y → Function[{x}, C[1] + x C[2] + 1. Cosh[1.8 x] + 1. Sinh[1.8 x]]}}
```

The **DSolve** treatment on the unauthorized second order equation does get the starting functions back.

```
solf = sol /. {C[1] → 0, C[2] → 0}
```

```
{{y → Function[{x}, 0 + x 0 + 1. Cosh[1.8 x] + 1. Sinh[1.8 x]]}}
```

Part (b)

```
Wronskian[{Cosh[1.8 x], Sinh[1.8 x]}, x]
```

```
1.8
```

Part (c) Solving the initial value part of the problem.

```
solivp = DSolve[{y''[x] == eqn, y[0] == 14.20, y'[0] == 16.38}, y, x]
```

```
{{y → Function[{x}, 13.2 + 14.58 x + 1. Cosh[1.8 x] + 1. Sinh[1.8 x]]}}
```

The test of the first initial value is successful for the proposed yellow solution.

```
13.2` + 14.579999999999998` x + 1.0000000000000002` Cosh[1.8` x] +
  1.0000000000000002` Sinh[1.8` x] /. x → 0
```

```
14.2
```

The yellow solution is differentiated.

```
D[13.2` + 14.579999999999998` x +
  1.0000000000000002` Cosh[1.8` x] + 1.0000000000000002` Sinh[1.8` x], x]
```

```
14.58 + 1.8 Cosh[1.8 x] + 1.8 Sinh[1.8 x]
```

The test of the second initial value is successful for the proposed yellow solution.

```
% /. x → 0
```

```
16.38
```

For reference, the text answer for the second order ODE is y” -3.24y = 0 and for the ivp sol’n it is 14.2 Cosh[1.8 x]+9.1 Sinh[1.8 x].